POISSON INTEGRAL FORMULA

It comes from the Cauchys integral formula. So we see it first. It basically says that the function (analytic) values inside a curve can be computed using the values of the function on the boundary.

THEOREM 0.1. *Let f be a function analytic inside and on a simple closed curve C, taken in the counter clockwise sense. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - z_0} ds.$$

Proof. For $\rho > 0$ sufficiently small, take C_ρ to the a circle with center at z_0 and radius ρ. Then by the Principle of deformation of paths,

$$\frac{1}{2\pi i} \int_C \frac{f(s)}{s - z_0} ds = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(s)}{s - z_0} ds$$

Now

$$\begin{aligned} \int_{C_\rho} \frac{f(s)}{s - z_0} ds &= \int_{C_\rho} \frac{f(s) - f(z_0) + f(z_0)}{s - z_0} ds \\ &= f(z_0) \int_{C_\rho} \frac{1}{s - z_0} ds + \int_{C_\rho} \frac{f(s) - f(z_0)}{s - z_0} ds \\ &= f(z_0) \cdot 2\pi\, i + \int_{C_\rho} \frac{f(s) - f(z_0)}{s - z_0} ds \end{aligned}$$

If we show that the second integral is zero, the proof will get over. To show it, take any $\epsilon > 0$. Since f is continuous at z_0, there exists a $\delta > 0$ such that $|f(s) - f(z_0)| < \varepsilon$ whenever $|s - z_0| < \delta$.

We choose our $\rho < \delta$. Hence for each s on C_ρ, we have $|f(s) - f(z_0)| < \varepsilon$. Hence

$$\left| \int_{C_\rho} \frac{f(s) - f(z_0)}{s - z_0} ds \right| \leq \int_{C_\rho} \frac{|f(s) - f(z_0)|}{|s - z_0|} ds \leq \varepsilon \int_{C_\rho} \frac{1}{\rho} ds = 2\pi\varepsilon.$$

Since it is true for every $\varepsilon > 0$, we get

$$\left| \int_C \frac{f(s)}{s - z_0} ds - f(z_0) \cdot 2\pi i \right| = 0,$$

which proves the Cauchy's integral formula. □

Now we move onto proving the Poisson integral formula. It gives the values of the real part of an analytic function inside a circle, using its values on the circle (multiplied by a kernal). It helps in solving many Dirichlet problems.

THEOREM 0.2. *Suppose $f(z)$ is a function analytic in a domain containing the circle C_0 with center at origin and radius $r_0 > 0$. Let the real part of $f(z)$ be $u(r,\theta)$ in polar cordinates. For $r < r_0$, let*

$$P(r_0, r, \phi - \theta) = \frac{{r_0}^2 - r^2}{{r_0}^2 + r^2 - 2r_0 r\cos(\phi - \theta)},$$

which is called the Poisson Kernal. Then

$$u(r,\theta) = \frac{1}{2\pi i}\int_0^{2\pi} P(r_0, r, \phi - \theta)u(r_0, \phi)d\phi, \quad r < r_0.$$

Proof. Suppose $f(z) = u(r,\theta) + i\,v(r,\theta)$. We know that for each z inside C_0 and z_1 outside C_0, by Cauchy's theorem and Cauchy's integral formula,

$$\int_{C_0}\frac{f(s)}{s - z_1}ds = 0 \quad \text{and} \quad \int_{C_0}\frac{f(s)}{s - z}ds = 2\pi\, i\cdot f(z).$$

Adding the two we get

$$f(z) = \frac{1}{2\pi i}\int_{C_0} f(s)(\frac{1}{s - z} - \frac{1}{s - z_1})ds$$

We choose z_1 in a particular way.

Given a point z inside a circle C, the inverse of z with respect to C is defined as the point z_1 lying on the same ray passing through z, which satisfy the condition $|z_1||z| = r^2$, where r is the radius of C. It is easy to observe that the inverse z_1 will be lying outside C because $|z_1| = \frac{r^2}{|z|} > r$, since $|z| < r$. For a z inside C_0, we choose z_1 to be the inverse of z to get the Poisson formula. With this z_1, we can write

$$f(z) = \frac{1}{2\pi i}\int_{C_0} f(s)(\frac{1}{s - z} - \frac{1}{s - z_1})ds$$

Here since s runs over C_0, we can write $s = r_0 e^{i\phi}$, $0 \leq \phi < 2\pi$. Then $ds = r_0\, i\, e^{i\phi}d\phi = is\,d\phi$. Substituting this in the integral,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi}(\frac{s}{s - z} - \frac{s}{s - z_1})f(s)\,d\phi$$

Since z_1 is also lying on the same ray containing z, and $|z_1| = \frac{{r_0}^2}{r}$, we can write $z_1 = \frac{{r_0}^2}{r}e^{i\theta} = \frac{{r_0}^2}{re^{-i\theta}} = \frac{s\overline{s}}{\overline{z}}$. Hence

$$\begin{aligned} f(z) &= \frac{1}{2\pi}\int_0^{2\pi}(\frac{s}{s - z} - \frac{s}{s - \frac{s\overline{s}}{\overline{z}}})f(s)\,d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}(\frac{s}{s - z} - \frac{\overline{z}}{\overline{s} - \overline{z}})f(s)\,d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}(\frac{s\overline{s} - z\overline{z}}{|s - z|^2})f(s)\,d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}(\frac{{r_0}^2 - r^2}{|s - z|^2})f(s)\,d\phi. \end{aligned}$$

That is,

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi}(\frac{{r_0}^2 - r^2}{|s - z|^2})f(r_0 e^{i\phi})\,d\phi, \quad \text{where } 0 < r < r_0.$$

This can be treated as the poisson integral formula in complex form. Using the triangle relation $a^2 = b_2 + c^2 - 2ab\cos A$ applied to the triangle formed by $0, s$ and z, we get $|s - z| = {r_0}^2 + r^2 - 2r_0 r\cos(\theta - \phi)$. Substituting this

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{({r_0}^2 - r^2)f(r_0 e^{i\phi})}{{r_0}^2 + r^2 - 2r_0 r\cos(\theta - \phi)}\Big)\,d\phi, \qquad \text{where } 0 < r < r_0$$

$$u(r,\theta) + i\,v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{({r_0}^2 - r^2)\big(u(r_0,\phi) + i\,v(r_0,\phi)\big)}{{r_0}^2 + r^2 - 2r_0 r\cos(\theta - \phi)}\Big)\,d\phi, \qquad \text{where } 0 < r < r_0$$

Collecting the real part and calling $P(r_0, r, \phi - \theta) = \frac{{r_0}^2 - r^2}{{r_0}^2 + r^2 - 2r_0 r\cos(\phi - \theta)}$, we get the Poisson integral formula as

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(r_0, r, \phi - \theta)u(r_0,\phi)\,d\phi, \qquad \text{where } 0 < r < r_0,$$

which is applicable to all harmonic functions $u(r,\theta)$, obtaining its values incide the circle ($r < r_0$) in terms of the values of $u(r_0,\phi)$ with the help of the kernal. □

COROLLARY 0.3. *Suppose the circle C_0 is the unit circle, then the Poisson integral formula becomes*

$$\begin{aligned} f(re^{i\theta}) &= \frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{(1 - r^2)f(e^{i\phi})}{1 + r^2 - 2r\cos(\theta - \phi)}\Big)\,d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} P(r, \phi - \theta)f(e^{i\phi})\,d\phi, \qquad \textit{where } 0 < r < r_0, \end{aligned}$$

Proof. Take $r_0 = 1$ in Poisson integral formula. □

COROLLARY 0.4. $\frac{1}{2\pi}\displaystyle\int_0^{2\pi} P(r_0, r, \phi - \theta)\,d\phi = 1$

Proof. Take $f = 1$ in Poisson integral formula. □