Partial Differential Equations for Engineering

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1 Introduction

In many practical problems we come across functions depending on more than one variable. For example, the vibration at a point of a stretched string depends on its distance from end points as well as time. Such functions are multivariable functions. That is, functions from $\mathbb{R}^2$ or, in general, $\mathbb{R}^n$ to $\mathbb{R}$.

Since the function values depends on more than one variable, the rate of change of the function values may be studied with respect to each one of the variables. This means that such functions may have the so called partial derivatives. We define it in the case of a two variable function.

Definition 1.1. *We say that $f : \mathbb{R}^2 \to \mathbb{R}$ is said to have a partial derivative with respect to x at a point x_0 if the limit*

$$\lim_{h\to 0} \frac{f(x_0 + h, y) - f(x_0, y)}{h}$$

exists. The derivative of f with respect to x is usually denoted by $\frac{\partial f}{\partial x}$.

The same definition can be extended to multivariable functions having any number of independent variables.

Now, as in the case of ordinary differential equations (ODE), one can consider equations having partial derivatives.

2 Preliminaries

In this section we consider partial differential equations (PDE) and related preliminary concepts.

Definition 2.1. *An equation which involves a dependent variable, more than one independent variables and the partial derivatives of the dependent variable with respect to the independent variables is called a PDE.*

The most general form of a PDE is

$$F(u, x, y, \dots, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \dots, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \dots) = 0$$

For example,

$$x\frac{\partial u}{\partial x} - y\frac{\partial^2 u}{\partial y^2} = x^3 + 4.$$

The order of the highest order derivative in the PDE is called the *order* of the PDE. In the above example the PDE is of order 2 since $\frac{\partial^2 u}{\partial x^2}$ is the highest order derivative appearing.

3 Formation of PDE

We know that the ODE corresponding to an equation is obtained by differentiating it and eliminating the arbitrary constants in the PDE. A first order PDE is formed by doing either of the following, after partial differentiation with respect to independent variables.

1. Eliminating arbitrary constants in the equation
2. Eliminating arbitrary functions in the equation.

Note that in the case when z is dependent variable and x, y are independent variables in PDE, we usually use the notations

$$\frac{\partial z}{\partial x} =: p, \frac{\partial z}{\partial y} =: q, \frac{\partial^2 z}{\partial x^2} =: r, \frac{\partial^2 z}{\partial x \partial y} =: s, \frac{\partial^2 z}{\partial y^2} =: t.$$

Example 3.1. Consider the equation $z = (x + a)(y + b)$.

Differentiating partially, we have

$$p = \frac{\partial z}{\partial x} = (y + b), \ q = \frac{\partial z}{\partial y} = (x + a)$$

Now eliminating a, b using the three equations, we form the PDE

$$z = pq.$$

Example 3.2. Consider the equation $z = f(u - v)$, where f is an arbitrary function.

Differentiating partially this equation, we have

$$p = \frac{\partial z}{\partial x} = f'(u - v) \times 1, \quad q = \frac{\partial z}{\partial y} = f'(u - v) \times -1.$$

Now, eliminating f (or f') we form the PDE: $p = -q$ or

$$p + q = 0.$$

4 Solution of First order PDE

In this section, we consider the first order PDEs and their solution. Here also, z stands for a dependent variable and x, y are independent variables, if not mentioned otherwise.

The general form of a first order PDE in two variables is

$$F(z, x, y, p, q) = 0.$$

There are many methods of solving first order PDEs: The ODE techniques, Lagrange's method for linear PDE, Charpit's method, Jacobi's method, method of separation of variables etc.

4.1 Method 1: Reducing PDE to ODE

Some PDEs can be reduced to the form of ODE and can be treated as ODEs while solving. For example, suppose that the PDE can be written as $\frac{\partial z}{\partial x} = f(x, y)$. Then since z has been differentiated partially with respect to x, we can reach back the solution by integrating it with respect to x, treating y as constant. This will solve such kind of a PDE. One should be careful to take the constant of integration as a function of y if the integration is by treating y as a constant.

Example 4.1. Consider the PDE $p = xy$.

Here z has been differentiated with respect to x, treating y as a constant to arrive at this PDE. Hence to solve it, one may integrate the PDE with respect to x, treating y as a constant. This gives the solution:

$$z = y\frac{x^2}{2} + C(y)$$

Note: This method can be applied to higher order equations also. For example, consider

$$\frac{\partial^n z}{\partial^m x \partial^k y} = f(x, y),$$

Here since z has been differentiated m times partially with respect to x and k times partially with respect to y, we can reach back the solution by direct integration k times with respect to y, treating x as constant and m times with respect to x, treating y as constant.

Example 4.2. Consider the PDE $\frac{\partial^2 z}{\partial x \partial y} = xy$.

Here z has been differentiated first with respect to y, treating x as a constant and then with respect to x, treating y as a constant to arrive at this PDE. That is

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = xy.$$

Hence to solve it, one may integrate the PDE first with respect to x, treating y as a constant and then with respect to y treating x as a constant.

$$\frac{\partial z}{\partial y} = y\frac{x^2}{2} + C_1(y)$$

and integrating a second time with respect to y,

$$z = \frac{x^2}{2}\frac{y^2}{2} + \int C_1(y)\, dy + C_2(x)$$

Another situation when we can consider PDE as ODE happens when the PDE contains only one independent variable explicitly, i.e., the other one is silent. Then one can treat it as an ODE and solve it using ODE methods.

Example 4.3. Consider the equation $\frac{\partial z}{\partial x} = xz$.

Here the variable y is silent. Hence it can be treated as the ODE $\frac{dz}{dx} = xz$. This can be made variable separable and solved to get

$$ln\, z = \frac{x^2}{2} + C(y).$$

Note: case defined above can be used for solving higher order PDEs also.

Example 4.4. Consider the PDE:

$$\frac{\partial^2 z}{\partial x^2} = 4z$$

Here the variable y is silent. Hence it can be treated as a second order linear ODE with constant coefficients. That is

$$(D^2 - 4)z = 0.$$

The roots of the auxiliary equation are 2, -2, real and distinct. Hence the solution will be

$$z = C_1(y)\, e^{2x} + C_2(y)\, e^{-2x}.$$

Note that the constants may contain y, as y is treated as a constant in the process.

4.2 Linear PDE

A linear first order PDE is a PDE which is linear in p and q. That is

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z),$$

where P, Q, R are functions of x, y, z is the form of a linear PDE of first order. In order to solve it, we use a method called *Lagrange's method.*

In Lagrange's method, we consider an *auxiliary equation*

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

and solve it to get two independent solutions, say $u = c_1$ and $v = c_2$. Then the solution to the linear PDE is given by

$$f(u,v) = 0$$

where f is an arbitrary function.

Example 4.5. Consider the PDE $xp + yq = z$.

The auxiliary equation is given by

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}.$$

Now taking the first two equations and solving using variable separable method, we get $ln\, x = ln\, y + ln\, c_1$, which becomes $\frac{x}{y} = c_1$. Hence we can take

$$u = \frac{x}{y}$$

.

Similarly, by taking the last two, we get $ln\, y = ln\, z + ln\, c_2$ which gives $\frac{y}{z} = c_2$. Hence we can take

$$v = \frac{y}{z}.$$

Thus the solution of the given PDE is $f(\frac{x}{y}, \frac{y}{z}) = 0$.

Note: Solution can also be $f(\frac{x}{y}, \frac{x}{z})$ or any such, as there are many pairs of independent solution to the auxiliary equation.

One may also use permissible combinations of the auxiliary equation. For example, since

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d} = \frac{la+mc}{lb+md},$$

we can also have many equivalent forms of the auxiliary equation. In the above Example 4.5,

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{d(x-y)}{x-y} = \frac{d(x-z)}{x-z} = \frac{d(y-z)}{y-z},$$

and we may get different solutions of the same PDE,

$$f(\frac{x-y}{x-z}, \frac{x-y}{y-z}) = 0 \quad \text{or} \quad f(\frac{x-y}{x-z}, \frac{x-z}{y-z}) = 0 \quad \text{etc.}$$

Sometimes all the three parts of the auxiliary equation are combined to find the two solutions. That is we may use

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

with suitable l, m and n. The simple choices for l, m and n are constants or x, y and z or $\frac{1}{x}, \frac{1}{y}$ and $\frac{1}{z}$ etc.

Example 4.6. Consider the PDE: $(bz - cy)p + (cx - az)q = ay - bx$.

The auxiliary equation is

$$\frac{dx}{bz - cy} = \frac{dy}{cx - az} = \frac{dz}{ay - bx} = \frac{ldx + mdy + ndz}{l(bz - cy) + m(cx - az) + n(ay - bx)}$$

If we choose $l = a, m = b, n = c$, we get

$$\frac{dx}{bz - cy} = \frac{dy}{cx - az} = \frac{dz}{ay - bx} = \frac{adx + bdy + cdz}{0},$$

which implies $d(ax + by + cz) = 0$, giving $ax + by + cz = c_1$.

Choosing $l = x, m = y, n = z$, we get

$$\frac{dx}{bz - cy} = \frac{dy}{cx - az} = \frac{dz}{ay - bx} = \frac{xdx + ydy + zdz}{0},$$

which implies $d(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}) = 0$, giving $x^2 + y^2 + z^2 = c_2$.

Thus the solution to the PDE is $f(ax + by + cz, x^2 + y^2 + z^2) = 0$.

4.3 Charpit's method for solution of PDE

This method is usually used for solving first order non linear PDE. We consider any first order PDE in two independent variables here, which is represented in the form

$$f(x, y, z, p, q) = 0.$$

As in the case of linear equation we form a set of subsidiary equations, and we find a relation between x, y, z, p, q by solving the subsidiary equation:

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dy}{f_y + qf_z}$$

Let this relation be given by

$$\phi(x, y, z, p, q) = 0.$$

By using this relation, find p and q from the PDE in terms of the other variables, say $p = F_1(x, y, z, q)$ and $q = F_2(x, y, z, p)$. Then

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy = F_1(x, y, z, q)\,dx + F_2(x, y, z, p)\,dy,$$

which on solving gives the solution z of the PDE.

Example 4.7. Consider the PDE: $p^2 + q^2 = 1$.

The auxiliary equation is

$$\frac{dx}{-2p} = \frac{dy}{-2q} = \frac{dz}{-2p^2 - 2q^2} = \frac{dp}{0} = \frac{dq}{0}$$

From the last two equations, we get $p = a$, a constant, and $q = b$, another constant.

Now since $p^2+q^2 = 1$, we have $a^2+b^2 = 1$, or in other words, $b = \sqrt{1 - a^2} \Rightarrow b = \sqrt{1 - a^2}$. Now substituting in the equation $dz = p\,dx + q\,dy$, we get

$$dz = a\,dx + \sqrt{1 - a^2}\,dy,$$

which on integration, gives the solution

$$z = ax + \sqrt{1 - a^2}y + C$$

Note: This type of a solution($z = ax + g(a)y + C$) is expected whenever the PDE is of the form $f(p, q) = 0$, that is, it does not contain any terms in x, y, z.

Example 4.8. Consider the PDE: $p^2z + q^2 - 4 = 0$.

The auxiliary equation is

$$\frac{dx}{2pz} = \frac{dy}{2q} = \frac{dz}{2p^2z + 2q^2} = \frac{dp}{-p^3} = \frac{dq}{-p^2q}$$

From the last two equations, we get

$$\frac{dp}{p} = \frac{dq}{q},$$

which gives $p = Cq$. Substituting in the PDE we can get p and q in terms of z.

$$p^2z + C^2p^2 - 4 = 0 \Rightarrow p = \frac{2}{\sqrt{z + C^2}}, \; q = \frac{2C}{\sqrt{z + C^2}}$$

Now substituting in the equation $dz = p\,dx + q\,dy$, we get

$$dz = \frac{2}{\sqrt{z + C^2}}\,dx + \frac{2C}{\sqrt{z + C^2}}\,dy,$$

$$\sqrt{z + C^2}\,dz = 2\,dx + 2C\,dy,$$

which on integration, gives the solution

$$\frac{(z + C^2)^{\frac{3}{2}}}{\frac{3}{2}} = 2x + 2Cy + C_1$$

In Charpits method also, we can use combinations of the subsidiary equation as in the case of Lagrange equation.

Example 4.9. Consider the PDE: $z^2 = xypq$.

The subsidiary equation is

$$\frac{dx}{xyq} = \frac{dy}{xyp} = \frac{dz}{2pqxy} = \frac{dp}{2zp - pqy} = \frac{dq}{2zq - pqx}$$

By taking combinations of the first and second, and then using combination of last two, we get:

$$= \frac{pdx + qdy}{2xypq} = \frac{xdp + ydq}{2xzp - pqxy + 2yzp - pqxy}$$

adding the two:

$$= \frac{pdx + qdy + xdp + ydq}{2z(xp + yq)} = \frac{d(xp + yq)}{2z(xp + yq)}.$$

Now equating this with the third item of the subsidiary equation:

$$\frac{dz}{2pqxy} = \frac{d(xp + yq)}{2z(xp + yq)} \Rightarrow \frac{dz}{z} = \frac{d(xp + yq)}{(xp + yq)}$$

Solving it, we get $ln\,z = ln\,(xp + yq) + ln\,C$, which gives

$$z = C(xp + yq)$$

Now for getting p in terms of other variables (in a simple form) make a substitution for yq from the PDE:

$$z = C(xp + \frac{z^2}{xp})$$

This gives a quadratic equation in p, solving which we get p:

$$p^2(cx^2) - p(xz) + cz^2 = 0 \Rightarrow p = \frac{xz + \sqrt{1 - 4C^2}xz}{2Cx^2} = \frac{z}{Kx}$$

where $K = \frac{2C}{1+\sqrt{1-4C^2}}$.

Substituting this in the PDE, we get $q = \frac{Kz}{y}$.

Now, substituting in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{z}{Kx}\,dx + \frac{Kz}{y}\,dy$$
$$\frac{dz}{z} = \frac{1}{Kx}\,dx + \frac{K}{y}\,dy$$
$$ln\,z = \frac{1}{K}\,ln\,x + K\,ln\,y + ln\,C_1$$

Hence the solution of the PDE is

$$z = C_1 x^{\frac{1}{K}} y^K.$$

4.4 Jacobi's method for solution of PDE

Here also we consider a first order non linear PDE, in general, of the form:

$$f(x, y, z, p, q) = 0.$$

It is supposed to have a solution of the form $U(x, y, z) = 0$. In that situation,

$$p = \frac{\partial z}{\partial x} = -\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial z}} = -\frac{U_1}{U_3} \text{ (,say)}$$
$$q = \frac{\partial z}{\partial y} = -\frac{\frac{\partial U}{\partial y}}{\frac{\partial U}{\partial z}} = -\frac{U_2}{U_3} \text{ (,say)}$$

Substituting these in the PDE:

$$F(x, y, z, U_1, U_2, U_3) = 0$$

Now form the subsidiary equation:

$$\frac{dx}{F_{U_1}} = \frac{dy}{F_{U_2}} = \frac{dz}{F_{U_3}} = \frac{dU_1}{-F_x} = \frac{dU_2}{-F_y} = \frac{dU_3}{-F_z}$$

From these equations and the PDE, find U_1, U_2 and U_3. Now

$$\begin{aligned} dU &= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \\ &= U_1 dx + U_2 dy + U_3 dz \end{aligned}$$

From this, we get the solution of the PDE, which is $U(x, y, z) = 0$.

Example 4.10. Consider the PDE: $p^2x + q^2y = z$.

Substituting $p == -\frac{U_1}{U_3}$ and $q == -\frac{U_2}{U_3}$, the equation becomes

$$\Big(-\frac{U_1}{U_3}\Big)^2 x + \Big(-\frac{U_2}{U_3}\Big)^2 y = z$$

which gives

$$xU_1{}^2 + yU_2{}^2 - zU_3{}^2 = 0$$

The auxiliary equation is

$$\frac{dx}{2xU_1} = \frac{dy}{2yU_2} = \frac{dz}{-2zU_3} = \frac{dU_1}{-U_1{}^2} = \frac{dU_2}{-U_2{}^2} = \frac{dU_3}{U_3{}^2}$$

From the first and fourth equations, we get

$$\frac{dx}{2xU_1} = \frac{dU_1}{-U_1{}^2} \Rightarrow \frac{dx}{2x} = \frac{dU_1}{-U_1} \Rightarrow U_1 = \frac{C_1}{\sqrt{x}}.$$

Similarly, $U_2 = \frac{C_2}{\sqrt{y}}$ and $U_3 = \frac{C_3}{\sqrt{z}}$.

Now substituting in the equation

$$\begin{aligned} dU &= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \\ &= \frac{C_1}{\sqrt{x}}dx + \frac{C_2}{\sqrt{y}}dy + \frac{C_3}{\sqrt{z}}dz, \end{aligned}$$

which on integration, yields the solution

$$U = \frac{C_1 x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{C_2 y^{\frac{1}{2}}}{\frac{1}{2}} + \frac{C_3 z^{\frac{1}{2}}}{\frac{1}{2}}.$$

Hence the solution to the PDE is

$$2C_1\sqrt{x} + 2C_2\sqrt{y} + 2C_3\sqrt{z} = 0.$$

4.5 Some special non linear PDE

4.5.1 Type I : $f(p, q) = 0$

This type of a PDE does not contain x, y, z terms. It can be shown that the solution to such a PDE is always of the form

$$z = ax + by + c,$$

where b has to be found in terms of a.

Example 4.11. Consider the PDE: $p + q = 1$.

Here the solution must be of the form

$$z = ax + by + c.$$

Then, differentiating we get $p = a$ and $q = b$. Then $a + b = 1$, on elimination. Thus $b = 1 - a$ and the solution to the PDE is

$$z = ax + (1 - a)y + c.$$

Note: Eventhough we have not derived the solution in Example 4.11, it is easy to very that the PDE formed from the answer is $p + q = 1$.

4.5.2 Type II : $f(p, q, z) = 0$

This type of a PDE does not contain x, y terms. It can be shown that the solution to such a PDE is always of the form

$$z = \phi(u), \qquad \text{where } u = x + ay.$$

and ϕ has to be found.

To find ϕ, we have $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$. Substituting p, q in $f(p, q, z) = 0$, we get $f(\frac{dz}{du}, a\frac{dz}{du}, z) = 0$. This is an ODE in u and z, and can be solved to get $z = \phi(u)$.

Example 4.12. Consider the PDE: $p\,(1 + q) = qz$.

Assume that the solution is

$$z = \phi(u), \qquad \text{where } u = x + ay.$$

Then, differentiating we get $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ Substituting in the given PDE, we get $1 + a\frac{dz}{du} = az$, which on solving gives

$$u = ln\,(c_1(az - 1)) + c_2$$

Thus the solution of the PDE is $x + ay = ln\,(az - 1) + c_2$

4.5.3 Type III : $z = px + qy + f(p, q)$

This type of a PDE is said to be in Clairut's form. The solution will be always of the form

$$z = ax + by + f(p, q).$$

It is easy to see that this is a solution to the given PDE by forming the corresponding PDE.

Example 4.13. Consider the PDE: $z = px + qy + p^2 + q^2$.

The solution is

$$z = ax + by + a^2 + b^2.$$

4.5.4 Type IV : $f(p, x) = g(q, y)$

Here the PDE does not contain z term explicitly and can be separated in terms of x and y along with their corresponding derivatives. Since x and y are independent variables separated to two parts, they must be constant, if they are to be equal. Hence

$$f(p, x) = g(q, y) = k$$

Now solve $f(x, p) = k$, to get p in terms of x and solve $g(y, q) = k$, to get q in terms of y. Let these be $p = g_1(x)$ and $q = g_2(y)$. Now since $dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$, we have

$$dz = g_1(x)\, dx + g_2(y)\, dy,$$

which on integrating gives the solution to the PDE.

Example 4.14. Consider the PDE: $q - p + x - y = 0$.

That is,

$$x - p = y - q = k$$

From $x - p = k$ and $y - q = k$, we get

$$p = x - k \quad \text{and} \quad q = y - k.$$

Then $dz = (x - k)dx + (y - k)dy$, which on integration gives the solution

$$z = \frac{x^2}{2} - kx + \frac{y^2}{2} - ky + c.$$

5 Second order PDE

In this section, we consider the next level PDEs, the second order PDE having two independent variables.

A second order PDE in two independent variables x, y has the general form:

$$F(z, x, y, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}) = 0$$

Also we use the notations U_x for $\frac{\partial U}{\partial x}$, U_{xy} for $\frac{\partial^2 U}{\partial x \partial y}$, ... when U is a function of x and y.

6 Classification of second order PDE

Suppose we write the second order PDE as

$$A\,\frac{\partial^2 z}{\partial x^2} + 2B\,\frac{\partial^2 z}{\partial x \partial y} + C\,\frac{\partial^2 z}{\partial y^2} = G\left(z, x, y, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right).$$

Then we classify the PDE in the following order:

The PDE is said to be:

i Elliptic if $AC - B^2 > 0$

ii Parabolic if $AC - B^2 = 0$

iii Hyperbolic if $AC - B^2 < 0$

Example 6.1. Consider the PDE:

$$x^2\, U_{xx} + xy\, U_{yy} + y^2\, U_{xy} = 0.$$

Here $A = x^2$, $B = \frac{y^2}{2}$ and $C = xy$ and so $AC - B^2 = y(x^3 - \frac{y^3}{4})$.

Hence it is

i Elliptic if $y > 0$ & $x^3 > \frac{y^3}{4}$ or if $y < 0$ & $x^3 < \frac{y^3}{4}$

ii Parabolic if $y = 0$ or $x^3 = \frac{y^3}{4}$

iii Hyperbolic if $y > 0$ & $x^3 < \frac{y^3}{4}$ or if $y < 0$ & $x^3 > \frac{y^3}{4}$.

7 Method of separation of variables

This method is adopted when the solution of the PDE happens to be in a product of two terms each including only a single independent variable. That is, in this method we assume that the solution is in the form

$$z(x, y) = X(x)\, Y(y),$$

where X is a function of x, not involving y and Y is a function of y, not involving x.

Then $\frac{\partial z}{\partial x} = X'Y$, $\frac{\partial z}{\partial y} = X\dot{Y}$, $\frac{\partial^2 z}{\partial x^2} = X''\,Y$, $\frac{\partial^2 z}{\partial x \partial y} = X'\dot{Y}$, $\frac{\partial^2 z}{\partial y^2} = X\ddot{Y}$.

Note that to distinguish the derivative with respect to second variable, we use 'dot' notation.

After finding these derivatives, substitute them in the PDE. Now separate it in to two sides so that the equation gets separated in its independent variables. Then it will be of the form:

$$\text{ODE}(X, x) = \text{ODE}(Y, y) = k$$

Now solve the equations $\text{ODE}(X, x) = k$ and $\text{ODE}(Y, y) = k$ using ODE methods and find the solutions $X(x)$ and $Y(y)$. Hence the solution to the given PDE will be

$$z = X(x)\, Y(y).$$

Example 7.1. Consider the PDE: $p = 2q + z$.

Assume that the solution is $z = XY$. Then $p = X'Y$ and $q = X\dot{Y}$. Substituting in the given PDE,

$$X'Y = 2\,X\dot{Y} + XY.$$

Separating to two sides:

$$X'\frac{1}{X} = \frac{1}{Y}(2\dot{Y} + Y) = k.$$

Now,

$$X' = kX \;\Rightarrow\; X = c_1 e^{kx}$$

and

$$2\dot{Y} + Y = kY \;\Rightarrow\; Y = c_2 e^{\frac{k-1}{2}y}.$$

Hence the solution to the PDE is $z = c\,e^{kx} e^{\frac{k-1}{2}y}$.

Now we consider three important PDEs; the Laplace equation, Wave equation and the Heat equation.

7.1 Laplace Equation

This is an important equation which appears frequently in the potential theory. We solve it by the method of separation of variables. Consider the Laplace equation in two variables

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0.$$

Assume that the solution is $U = X(x)\,Y(y)$. Now substituting in the Laplace equation,

$$X''Y + X\,\ddot{Y} = 0.$$

Separating

$$\frac{X''}{X} = -\frac{\ddot{Y}}{Y} = k.$$

There are three possible cases:

case 1: $k = 0$

In this case,

$$\frac{X''}{X} = 0 \;\Rightarrow X'' = 0 \;\Rightarrow X = c_1 x + c_2$$

and

$$-\frac{\ddot{Y}}{Y} = 0 \;\Rightarrow \ddot{Y} = 0 \;\Rightarrow Y = c_3 y + c_4.$$

Hence the solution in this case is

$$U(x, y) = (c_1 x + c_2)(c_3 y + c_4).$$

case 2: $k > 0$

We take $k = \alpha^2$. Then solving the first part,

$$\frac{X''}{X} = \alpha^2 \;\Rightarrow X'' = \alpha^2 X \;\Rightarrow X = c_1\, e^{\alpha\, x} + c_2\, e^{-\alpha\, x}$$

and

$$-\frac{\ddot{Y}}{Y} = \alpha^2 \;\Rightarrow \ddot{Y} = -\alpha^2\, Y \;\Rightarrow Y = e^{0x}(c_3\, cos\, \alpha y + c_4\, sin\, \alpha y).$$

Hence the solution is

$$U(x, y) = (c_1\, e^{\alpha\, x} + c_2\, e^{-\alpha\, x})(c_3\, cos\, \alpha y + c_4\, sin\, \alpha y).$$

case 3: $k < 0$

We take $k = -\alpha^2$. Then solving the first part,

$$\frac{X''}{X} = -\alpha^2 \;\Rightarrow X'' = -\alpha^2 X \;\Rightarrow X = e^{0x}(c_1 \cos \alpha\, x + c_2 \sin\alpha\, x)$$

and

$$-\frac{\ddot{Y}}{Y} = -\alpha^2 \;\Rightarrow \ddot{Y} = \alpha^2 Y \;\Rightarrow Y = c_3\, e^{\alpha y} + c_4\, e^{-\alpha y},$$

giving the solution

$$U(x, y) = (c_1 \cos \alpha x + c_2 \sin \alpha x)(c_3\, e^{\alpha\, y} + c_4\, e^{-\alpha\, y}).$$

7.2 Wave Equation - Formation

We consider the vibrations of a stretched string, tied to both ends. Without loss of generality, assume that the string is kept parallel to X-axis and the length of the string is L. If U denotes the vertical displacement of a point on the string (on vibration), it depends on two quanities:

1. the distance of the point from the left end (denoted by x)
2. time elapsed after starting the vibration (denoted by t)

Hence

$$U = U(x, t).$$

It is clear that the end points will not vibrate at any instant of time. Mathematically,

$$U(0, t) = U(L, t) = 0 \text{ for all } t > 0.$$

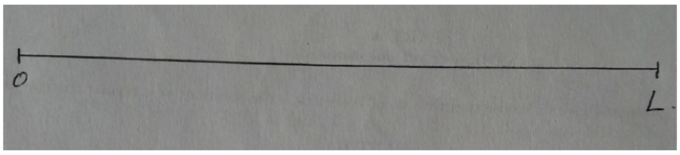

Figure 1: String before motion

In order to simplify the modeling, we make certain assumptions.

- Vibration happens only in one dimension and each point of the string can move only up and down (no horizontal motion happens).
- The string is homogenous. That is the mass per unit lengh (linear density ρ) is the same at any position of the string.
- String is perfectly elastic and is not resistant to bending.
- The tension at each point of the string is so large that the weight of the string can be negleted.
- The displacement and slopes are so small that their higher powers can be negleted.

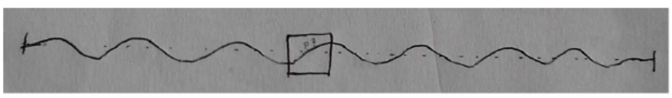

Figure 2: String during motion

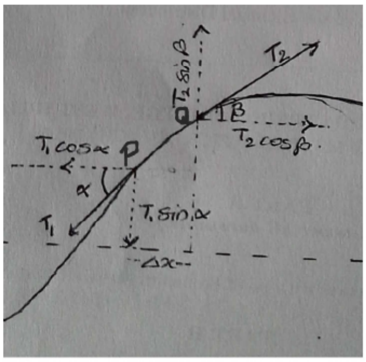

Figure 3: Enlarged portion PQ

To model the motion, we take a point P at a distance x from the left end and consider another adjascent point Q, at a small distance $\triangle\, x$ apart from P. We first find the equation of motion of the arc PQ and then make $Q \to P$ (or equivalently $\triangle\, x \to 0$) at the end in order to get the equation of motion at the point P.

Let the angles that the tangents at P and Q makes with X axis be α and β respectively. Suppose that the magnitudes of tensions at P and Q be T_1 and T_2 respectively, acting in the tangential direction. Then the horizontal components of the tensions at P and Q are respectively $T_1\, cos\alpha$ and $T_2\, cos\beta$ (to opposite directions).

Since there is no horizontal vibration, we have

$$T_1\, cos\alpha = T_2\, cos\beta = T, \text{ (say)} \tag{1}$$

But we know that the string vibrates vertically. This happens due to the force in that direction, which is the vertical components of the tensions at the points P and Q. Thus the effective force on the arc PQ is given by

$$F = T_2\, sin\, \beta - T_1\, sin\, \alpha \tag{2}$$

But Newton's law of motion says that the force on PQ should be equal to the product of mass and acceleration. Since the linear density is ρ and the length of PQ is $\triangle\, x$, it mass is $\triangle\, x\rho$. Hence

$$F = \triangle\, x\, \rho\, \frac{\partial^2 U}{\partial t^2} \tag{3}$$

From (2) and (3), we have the equation of motion of the arc PQ:

$$T_2\, sin\, \beta - T_1\, sin\, \alpha = \triangle\, x\, \rho\, \frac{\partial^2 U}{\partial t^2} \tag{4}$$

Dividing this equation by the product of $T\ \triangle\, x$ (and using $T_2\, cos\beta = T_1\, cos\alpha = T$),

$$\frac{tan\, \beta - tan\, \alpha}{\triangle\, x} = \frac{\rho}{T}\, \frac{\partial^2 U}{\partial t^2} \tag{5}$$

Now $tan\, \beta$ is the slope at Q, that is $\frac{\partial U}{\partial x}$ at $x + \triangle\, x$ and $tan\, \alpha$ is the slope $\frac{\partial U}{\partial x}$ at x. Applying it,

$$\frac{\frac{\partial U}{\partial x}|_{(x+\triangle x)} - \frac{\partial U}{\partial x}|_{(x)}}{\triangle\, x} = \frac{\rho}{T}\, \frac{\partial^2 U}{\partial t^2} \tag{6}$$

This is another form of equation of motion of the arc PQ. Now to get the equation of motion at the point P, we take the limiting situation when $Q \to P$ or $\triangle\, x \to 0$. That is

$$\lim_{\triangle x \to 0} \frac{\frac{\partial U}{\partial x}|_{(x+\triangle x)} - \frac{\partial U}{\partial x}|_{(x)}}{\triangle\, x} = \lim_{\triangle x \to 0} \frac{\rho}{T}\, \frac{\partial^2 U}{\partial t^2} \tag{7}$$

But then the LHS gives the partial derivative of $\frac{\partial U}{\partial x}$ with respect to x and the RHS will not have any change since it does not contain any $\triangle\, x$. Thus

$$\frac{\partial^2 U}{\partial t^2} = \frac{T}{\rho}\,\frac{\partial^2 U}{\partial x^2}$$

is the equation of vibration, known as the *wave equation.* Since ρ and T are strictly positive, we usually take $\frac{T}{\rho}$ as c^2 (or, sometimes $\frac{\rho}{T}$ as c^2). Thus the wave equation is

$$\frac{\partial^2 U}{\partial t^2} = c^2\,\frac{\partial^2 U}{\partial x^2}.$$

Apart from the mentioned conditions

1. The left end does not move at any time. That is, $U(0,t) = 0$ for all t
2. The right end does not move at any time. That is $U(L,t) = 0$ for all t,

we make two more assumptions usually, while modeling the wave equation:

3. The initial position of the string is known. That is $U(x,0) = f(x)$, say.
4. The initial velocity of the string is known. That is $U_t(x,0) = g(x)$, say.

7.3 Wave Equation - solution

Consider the wave equation

$$\frac{\partial^2 U}{\partial t^2} = c^2\,\frac{\partial^2 U}{\partial x^2}$$

with the conditions $U(0,t) = 0 = U(L,t)$ for all t, $U(x,0) = f(x)$ and $U_t(x,0) = g(x)$.

Here we solve the wave equation by separation of variables method. Assume that the solution is $U(x,t) = X(x)\,Y(t)$. Substituting in wave equation,

$$X\,\ddot{Y} = c^2\,X''Y$$

Separating

$$\frac{X''}{X} = \frac{\ddot{Y}}{c^2 Y} = k.$$

There are three possible cases: $k = 0$, $k > 0$ and $k < 0$.

[Before going in to these cases, we observe that the intial condition $U(0,t) = 0$ implies $X(0)Y(t) = 0$, forcing $X(0) = 0$, since if $Y(t) = 0$, there will not be any vibration as $U = 0$. Similarly $U(L,t) = 0$ implies that $X(L) = 0$.]

case 1: $k = 0$

In this case,

$$\frac{X''}{X} = 0 \;\Rightarrow X'' = 0 \;\Rightarrow X = c_1 x + c_2.$$

Now using the condition $X(0) = 0$, we get

$$c_2 = 0.$$

Thus $X(x) = c_1 x$. Now $X(L) = 0$ implies

$$c_1 = 0$$

(as L is strictly positive). Thus

$$X(x) = 0,$$

giving $U = 0$, leading to an uninteresting situation - there is no vibration at all.

case 2: $k > 0$, say $k = \alpha^2$, where $\alpha > 0$.

$$\frac{X''}{X} = \alpha^2 \;\Rightarrow X'' = \alpha^2\, X \;\Rightarrow X = c_1\, e^{\alpha\, x} + c_2\, e^{-\alpha\, x}$$

Now using the condition $X(0) = 0$, we get

$$c_1 + c_2 = 0. \tag{8}$$

Now $X(L) = 0$ implies

$$c_1\, e^{\alpha L} + c_2\, e^{-\alpha L} = 0 \tag{9}$$

(8) $\times\, e^{\alpha L}$ – (9) gives

$$c_1 = 0$$

and substituting this c_1 in (8) gives

$$c_2 = 0.$$

Thus

$$X(x) = 0,$$

giving $U = 0$, leading to an uninteresting situation - there is no vibration at all.

case 3: $k < 0$, say $k = -\alpha^2$, where $\alpha > 0$.

$$\frac{X''}{X} = -\alpha^2 \;\Rightarrow X'' = -\alpha^2\, X \;\Rightarrow X(x) = c_1\, cos\, \alpha\, x + c_2\, sin\alpha\, x$$

Now using the condition $X(0) = 0$, we get

$$c_1 = 0, \tag{10}$$

making $X(x) = c_2\, sin\alpha\, x$. Now $X(L) = 0$ implies

$$c_2\, sin\alpha\, L = 0 \tag{11}$$

Since $c_2 \neq 0$ (if $c_2 = 0$, $X = 0 \Rightarrow U = 0$), we must have $\alpha L = n\pi$, for each $n = 0, 1, 2, \ldots$ (negative values of n will repeat the values of α^2).

Thus $\alpha = \frac{n\pi}{L}$ and hence

$$X(x) = c_2\, sin \frac{n\pi}{L} x, \text{ for each } n.$$

Now we find the solution for Y. For this,

$$\ddot{Y} = -\alpha^2 c^2 Y,$$

giving complex conjugate roots ($\pm\, i\, \alpha\, c\, = \pm\, i\, \frac{n\pi\, c}{L}$) for auxiliary equation and the solution

$$Y(t) = b_1 cos\left(\frac{n\pi c}{L}\, t\right) + b_2 sin\left(\frac{n\pi c}{L}\, t\right).$$

Thus the solution is given by

$$U(x,t) = \left[b_1 cos\left(\frac{n\pi c}{L}\, t\right) + b_2 sin\left(\frac{n\pi c}{L}\, t\right) \right]\; c_2\, sin\left(\frac{n\pi}{L} x\right), \text{ for each } n.$$

But since c_2, b_1, b_2 are arbitrary, they will also change as n vary. We use A_n, B_n for constants $b_1\, c_2$ and $b_2\, c_2$ respectively. Now

$$U_n(x,t) = \left[A_n cos\left(\frac{n\pi c}{L}\, t\right) + B_n sin\left(\frac{n\pi c}{L}\, t\right) \right]\; sin\left(\frac{n\pi}{L} x\right), \text{ for each } n.$$

But we know that for a homogenous equation, the sum of the solutions is also a solution. Here it turns out that the infinite sum of these expressions form the general solution to the wave equation. Hence

$$U(x,t)=\sum_{n=0}^{\infty}\left(A_n\,cos\,\frac{n\pi c}{L}t+B_n\,sin\,\frac{n\pi c}{L}t\right)\,sin\frac{n\pi}{L}x \tag{12}$$

is the general solution to the wave equation. In order to find the arbitrary constants A_n and B_n, we need to apply the second and third boundary conditions.

Now, $U(x,0)=f(x)$ gives

$$f(x)=\sum_{n=0}^{\infty}A_n\,sin\frac{n\pi}{L}x,$$

which is actually the sine series expansion of $f(x)$. From theory of fourier series, it follows that

$$A_n=\frac{2}{L}\int_0^L f(x)sin\frac{n\pi}{L}x\,dx. \tag{13}$$

Similarly, $U_t(x,0)=g(x)$ gives

$$g(x)=\sum_{n=0}^{\infty}B_n\,\frac{n\pi\,c}{L}\,sin\frac{n\pi}{L}x,$$

which is actually the sine series expansion of $g(x)$. From theory of Fourier series, it follows that

$$B_n\,\frac{n\pi\,c}{L}\;=\frac{2}{L}\int_0^L g(x)sin\frac{n\pi}{L}x\,dx,$$

which gives the constant

$$B_n=\frac{2}{c\,n\pi}\int_0^L g(x)sin\frac{n\pi}{L}x\,dx. \tag{14}$$

Thus equation (12) togethor with (13) and (14) gives the complete solution of wave equation.

7.3.1 Problems

1. A tightly streched string of lenth L is initially at rest in equilibrium position. If it is set vibrating by giving to each its points a velocity $\lambda x(L-x)$, find the displacement of string at any distance x from one end at any time t.

   Solution: To solve this, let us consider the solution of wave equation

   $$U(x,t)=\sum_{n=0}^{\infty}\left(A_n\,cos\,\frac{n\pi c}{L}t+B_n\,sin\,\frac{n\pi c}{L}t\right)\,sin\frac{n\pi}{L}x$$

   Now the string is at rest in its equilibrium position initially. This means

   $$U(x,0)=0=f(x).$$

   So

   $$\sum A_n\,sin\,\frac{n\pi}{L}x=0$$

   This implies

   $$A_n=\frac{2}{L}\int 0\,dx=0$$

Hence

$$U(x,t) = \sum_{n=0}^{\infty} B_n \, sin\, \frac{n\pi c}{L}\, t \, sin \frac{n\pi}{L} x \tag{15}$$

Now, the initial velocity at each point is given by

$$U_t(x,0) = \lambda x(L-x) = g(x)$$

This means that

$$\lambda x(L-x) = \sum_{n=0}^{\infty} B_n \, \frac{n\pi c}{L} \, sin \frac{n\pi}{L} x \tag{16}$$

Using Fourier sine series,

$$\frac{n\pi c}{L} B_n = \frac{2}{L} \int_0^L \lambda x(L-x) \, sin \, \frac{n\pi}{L} x \, dx$$

That is

$$\begin{aligned} B_n &= \frac{2}{n\pi c} \int_0^L \lambda x(L-x) \, sin \, \frac{n\pi}{L} x \, dx \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{8\,\lambda L^3}{n^4 \pi^4 c} & \text{otherwise} \end{cases} \end{aligned} \tag{17}$$

Thus (15) with (17) gives the complete solution to the problem.

7.4 The D'Alemberts Solution to Wave Equation

If we observe the solution to wave equation (12), we can see that the solution can be written in terms of $x + ct$ and $x - ct$. We solve the wave equation

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}$$

with the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ using D'Alemberts method.

We first define two new variables $v = x + ct$ and $w = x - ct$. Then

$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial w}{\partial x} = 1, \quad \frac{\partial v}{\partial t} = c, \quad \frac{\partial w}{\partial t} = -c.$$

Since $x = \frac{v+w}{2}$ and $t = \frac{v-w}{2c}$, we can treat x and t as functions of v and w. Thus $U(x,t)$ can be treated as a function of v and w. Now

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial U}{\partial w}\frac{\partial w}{\partial x} = U_v + U_w$$

and

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) &= \frac{\partial}{\partial v}\left(\frac{\partial U}{\partial x}\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial w}\left(\frac{\partial U}{\partial x}\right)\frac{\partial w}{\partial x} \\ &= \frac{\partial}{\partial v}\left(U_v + U_w\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial w}\left(U_v + U_w\right)\frac{\partial w}{\partial x} \\ &= U_{vv} + 2U_{vw} + U_{ww}. \end{aligned}$$

In a similar way,

$$\frac{\partial U}{\partial t} = c\left(U_v - U_w\right)$$

and

$$\frac{\partial^2 U}{\partial t^2} = c^2 \, \left(U_{vv} - 2U_{vw} + U_{ww}\right).$$

Substituting in wave equation, we get

$$U_{vw} = 0,$$

which is the wave equation in the new variables.

To solve it, it is enough to integrate it, first with respect to v and then with respect to w, partially. Thus the solution is

$$U(x,t) = \phi(x+ct) + \psi(x-ct),$$

where ϕ and ψ are arbitrary functions to be evaluated.

Using the condition $U(x,0) = f(x)$, we get

$$\phi(x) + \psi(x) = f(x) \tag{18}$$

Now the second condition $U_t(x,0) = g(x)$ implies

$$\phi'(x) - \psi'(x) = \frac{1}{c} g(x) \tag{19}$$

integrating (19) from x_0 to x, we get

$$\phi(x) - \psi(x) = \phi(x_0) - \psi(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)\,ds \tag{20}$$

Adding (18) and (20),

$$\phi(x) = \frac{1}{2}\left[f(x) + \phi(x_0) - \psi(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)\,ds\right]$$

and hence

$$\phi(x+ct) = \frac{1}{2}\left[f(x+ct) + \phi(x_0) - \psi(x_0) + \frac{1}{c}\int_{x_0}^{x+ct} g(s)\,ds\right] \tag{21}$$

Similarly on doing (18) $-$ (20), we get

$$\psi(x-ct) = \frac{1}{2}\left[f(x-ct) - \phi(x_0) + \psi(x_0) - \frac{1}{c}\int_{x_0}^{x-ct} g(s)\,ds\right] \tag{22}$$

Thus

$$U(x,t) = \phi(x+ct) + \psi(x-ct) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds,$$

which is called the D'Alemberts solution to wave equation.

7.5 Heat Equation - Formation

Here we consider the conduction of heat through a uniform bar of length L. The temperature at a cross sectional area of the bar, denoted by U, is a function of temperature t and its position (we measure it from one end point) x. That is, $U = U(x,t)$.

In order to smoothen the modeling procedure, we make some assumptions:

1. The bar is of uniform area of cross section, say A.
2. The sides of the bar are insulated so that heat will not escape along the sides.
3. The temperature at each point of a cross sectional area are the same.

Consider a cross sectional area P at a distance x. Let Q_1 be the heat flowing into this cross sectional area. Now, if κ is the conductivity of the material and A is the cross sectional area, then the amount of heat entering into P is given by

$$Q_1 = -\kappa A \, \frac{\partial U}{\partial x}_{\text{(at } x)} \tag{23}$$

Consider an adjascent cross sectional area Q, at a distance $\triangle\, x$ from P. Then the heat leaving the cross section Q is given by

$$Q_2 = -\kappa A \, \frac{\partial U}{\partial x}_{\text{(at } x+\triangle x)} \tag{24}$$

Since the sides are assumed to be insulated, the difference amount [(23) - (24)] is the amount of heat stored inside the portion PQ. That is the heat stored inside PQ is given by

$$H = \kappa A \left(\frac{\partial U}{\partial x}_{\text{(at } x+\triangle x)} - \frac{\partial U}{\partial x}_{\text{(at } x)} \right) \tag{25}$$

Now if S is the *specific heat* (amount of heat required to raise the temperature of the material of unit mass by 1^0C) of the material, then the heat stored in PQ is given by

$$H = S\, m \, \frac{\partial U}{\partial t}, \tag{26}$$

where m is the mass of PQ. Now, if ρ is the density of the material, $m = \rho A \,\triangle\, x$ (since $A \,\triangle\, x$ is the volume of PQ) and equation (26) becomes

$$H = S\, \rho A \,\triangle\, x \, \frac{\partial U}{\partial t}, \tag{27}$$

But both the equations (25) and (27) give the heat stored inside PQ. Equating, we get

$$S\, \rho A \,\triangle\, x \, \frac{\partial U}{\partial t} = \kappa A \left(\frac{\partial U}{\partial x}_{\text{(at } x+\triangle x)} - \frac{\partial U}{\partial x}_{\text{(at } x)} \right)$$

Or

$$\frac{\partial U}{\partial t} = \frac{\kappa}{S\rho} \left(\frac{\frac{\partial U}{\partial x}|_{x+\triangle x} - \frac{\partial U}{\partial x}|_{x}}{\triangle\, x} \right),$$

which gives the heat equation corresponding to the portion PQ. Now to get the heat equation at the cross sectional area P, make $Q \to P$ or equivalently, $\triangle\, x \to 0$:

$$\frac{\partial U}{\partial t} = \frac{\kappa}{S\rho} \lim_{\triangle x \to 0} \left(\frac{\frac{\partial U}{\partial x}|_{x+\triangle x} - \frac{\partial U}{\partial x}|_{x}}{\triangle\, x} \right),$$

Thus the heat equation is

$$\frac{\partial U}{\partial t} = \frac{\kappa}{S\rho} \left(\frac{\partial^2 U}{\partial x^2} \right),$$

Since κ, S, ρ are nonnegative, we abreviate $\frac{\kappa}{S\rho}$ as c^2. Thus the heat equation is

$$\frac{\partial U}{\partial t} = c^2 \, \frac{\partial^2 U}{\partial x^2}.$$

7.6 Heat Equation - Solution

We solve heat equation by method of separation of variables. For this, we assume that

$$U(x,t) = X(x)T(t).$$

Substituting this (derivatives of this) in the heat equation, we get

$$X\dot{T} = c^2 X''T$$

which gives

$$\frac{X''}{X} = \frac{\dot{T}}{c^2 T} = k$$

Here k is a real number. There are three situations:

Case 1: $k = 0$

Consider $\frac{X''}{X} = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = C_1 x + C_2$.

Now $\frac{\dot{T}}{c^2 T} = 0 \quad \Rightarrow \dot{T} = 0 \quad \Rightarrow T(t) = C_3$. Thus the solution in this case is:

$$U(x,t) = (C_1 x + C_2)C_3 \quad \Rightarrow \quad U(x,t) = Ax + B$$

Case 2: $k > 0$, say $k = \alpha^2$.

Consider $X'' = \alpha^2 X$. The auxiliary equation is $m^2 - \alpha^2 = 0$, giving distinct real roots, $m = \alpha,\ -\alpha$. Then X is given by

$$X(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$$

Now $\frac{\dot{T}}{c^2 T} = \alpha^2 \quad \Rightarrow \dot{T} = \alpha^2 c^2 T \quad \Rightarrow T(t) = C_3 e^{\alpha^2 c^2 t}$.

Hence the solution is given by

$$U(x,t) = (C_1 e^{\alpha x} + C_2 e^{-\alpha x})C_3 e^{\alpha^2 c^2 t} = (Ae^{\alpha x} + Be^{-\alpha x})e^{\alpha^2 c^2 t}$$

Case 3: $k < 0$, say $k = -\alpha^2$.

Consider $X'' = -\alpha^2 X$. The auxiliary equation is $m^2 + \alpha^2 = 0$, giving complex conjugate roots, $m = i\,\alpha,\ -i\,\alpha$. Then X is given by

$$X(x) = C_1 cos\,\alpha x + C_2 sin\,\alpha x$$

Now $\frac{\dot{T}}{c^2 T} = -\alpha^2 \quad \Rightarrow \dot{T} = -\alpha^2 c^2 T \quad \Rightarrow T(t) = C_3 e^{-\alpha^2 c^2 t}$.

Hence the solution is given by

$$U(x,t) = (C_1 cos\,\alpha x + C_2 sin\,\alpha x)C_3 e^{-\alpha^2 c^2 t} = (A\,cos\,\alpha x + B\,sin\,\alpha x)e^{-\alpha^2 c^2 t}$$

Note that if we impose the boundary conditions $U(0,t) = 0 = U(L,t)$, here also Case 3 is the only case of interest, as in the case of wave equation.

7.6.1 Problems

1. A rod of lenth L with insulated sides is initially at a uniform temperature U_0. Its ends are suddenly cooled to $0^0 C$ and are kept at that temperature. Find the temperature function.

   Solution: The heat equation is $U_t = c^2 U_{xx}$. Since $U(0,t) = 0 = U(L,t)$, the only case of interest is when $k < 0$ or $k = -\alpha^2$. In this case the solution is given by

   $$U(x,t) = (A\,cos\,\alpha x + B\,sin\,\alpha x)e^{-\alpha^2 c^2 t}$$

The conditions given are $U(0,t) = 0 = U(L,t)$ and $U(x,0) = U_0$. As in the case of wave equation the first condition imply $X(0) = 0$, which means

$$0 = A\, e^{-\alpha^2 c^2 t} \quad \Rightarrow A = 0$$

Thus

$$U(x,t) = (B\, sin\, \alpha x) e^{-\alpha^2 c^2 t}$$

Now the second condition gives

$$0 = B\, sin\, \alpha L\, e^{-\alpha^2 c^2 t} \quad \Rightarrow sin\, \alpha L = 0$$

which happens only if $\alpha = \frac{n\pi}{L}$ for each $n = 0, 1, 2, \ldots$. Thus the solution becomes

$$U_n(x,t) = (B_n\, sin\, \frac{n\pi}{L} x) e^{-\alpha^2 c^2 t} \text{ for each } n.$$

and the general solution is obtained by summing up:

$$U(x,t) = \sum (B_n\, sin\, \frac{n\pi}{L} x) e^{-\alpha^2 c^2 t} \tag{28}$$

Imposing the third condition $U(x,0) = U_0$, we get

$$U_0 = \sum (B_n\, sin\, \frac{n\pi}{L} x) e^0,$$

which is a sine series. Hence B_n are given by

$$\begin{aligned} B_n &= \frac{2}{L} \int_0^L U_0 sin \left(\frac{n\pi x}{L} \right) dx \\ &= \frac{2\, U_0}{n\pi} \left[1 - (-1)^n \right]. \end{aligned} \tag{29}$$

Thus (27) with (28) gives the solution to the problem. That is

$$U(x,t) = \sum \frac{2\, U_0}{n\pi} \left[1 - (-1)^n \right] \, sin\, (\frac{n\pi}{L} x)\, e^{-\alpha^2 c^2 t}$$

is the required solution.

2. The ends A and B of a rod $20cm$ long have the temperature at $30^0 C$ and $80^0 C$ until steady-state prevails. The temperature of the ends are changed to $40^0 C$ and $60^0 C$ respectively. Find thetemperature distribution in the rod at time t.

   Solution: This problem has two stages. First step is heating/cooling the rod in such a way that the rod reaches in a steady state (that is a state in which temperature does not vary further; thereby having no dependance on time). The second stage is giving further heat/cooling so that the ends are changed to non-zero, constant temperatures. So we need to do the problem in two stages.

   Stage 1: Making the rod to a steady state in which the ends are at $30^0 C$ and $80^0 C$. Since it is a steady state, $\frac{\partial U}{\partial t} = 0$. Now from heat equation, $U_t = c^2 U_{xx}$, we get $U_{xx} = 0$ also. This means

$$X\dot{T} = 0 \text{ and } X'' Y = 0 \Rightarrow \dot{T} = 0 \text{ and } X'' = 0 \Rightarrow T = C_1 \text{ and } X = C_2 x + C_3.$$

   So the temperature in steady state is (always) given by the equation:

$$U(x,t) = ax + b \qquad \text{(taking } C_1 C_2 = a \text{ and } C_1 C_3 = b)$$

To find a and b, apply the boundary conditions; that is $U(0,t) = 30$ and $U(20,t) = 80$. This gives $a = \frac{5}{2}$ and $b = 30$. Thus the temperature distribution at steady state is given by

$$U(x,t) = \frac{5}{2}x + 30.$$

Stage 2: From steady state to another state where the temperatures at end points are made to 40^0C and 60^0C.

Note that here we are starting from steady state. That is our initial temperature distribution (at $t = 0$) is given by $U(x,0) = \frac{5}{2}x + 30$ for $0 \leq x \leq 20$. Now we are making the rod to reach to a temperature 40^0C and 60^0C at the end points.

Apart from the heating/cooling effect supplied from outside, the temperature distribution is also affected by the existing temperature distribution prevailing at the steady state. Hence our required temperature function $U(x,t)$ will have two components: one corresponding to the prevailing steady state (which is a function of x only and will be denoted by $U_1(x)$) and the second one due to the temperature given from outside (which will be a function of x as well as t, denoted as $U_2(x,t)$). Thus

$$U(x,t) = U_1(x) + U_2(x,t)$$

is the present temperature function required in the problem. Here the boundary condition given are $U(0,t) = 40$, $U(20,t) = 60$ and $U(x,0) = \frac{5}{2}x + 30$.

We need to find U_1 and U_2. Since $U_1(x)$ does not depend on t, $U_{1t} = 0$ and from heat equation we get $U_{1xx} = 0$ and so $U_1(x) = ax + b$ as in the case of steady state. Again we need the constants a and b. For this, use the conditions $U_1(0) = 40$ and $U_1(20) = 60$, giving $a = 1$ and $b = 40$. Thus

$$U_1(x) = x + 40$$

This forces the second temperature function U_2 to have 0^0C at both end points, making in to the standard problem discussed in the previous example. To see this,

$$U(x,t) = x + 40 + U_2(x,t) \Rightarrow U_2(x,t) = U(x,t) - x - 40$$

Now $U_2(0,t) = U(0,t) - 0 - 40 = 40 - 40 = 0$ and $U_2(20,t) = U(20,t) - 20 - 40 = 60 - 60 = 0$. Now, $U_2(x,0) = U(x,0) - x - 40 = \frac{3x}{2} - 10$.

Hence we can find the function $U_2(x,t)$ as in the previous problem using the wave equation with initial conditions

$U_2(0,t) = 0$, $U_2(20,t) = 0$ and $U_2(x,0) = \frac{3}{2}x - 10$.

Think of U as U_2 in the following:

The heat equation is $U_t = c^2 U_{xx}$. Since $U(0,t) = 0 = U(20,t)$, the only case of interest is when $k < 0$ or $k = -\alpha^2$. In this case the solution is given by

$$U(x,t) = (A\,cos\,\alpha x + B\,sin\,\alpha x)e^{-\alpha^2 c^2 t}$$

The conditions given are $U(0,t) = 0 = U(20,t)$ and $U(x,0) = \frac{5}{2}x + 30$. As in the case of wave equation the first condition imply $X(0) = 0$, which means

$$0 = A\,e^{-\alpha^2 c^2 t} \quad \Rightarrow A = 0$$

Thus

$$U(x,t) = (B\,sin\,\alpha x)e^{-\alpha^2 c^2 t}$$

Now the second condition gives

$$0 = B\, sin\, \alpha L\, e^{-\alpha^2 c^2 t} \quad \Rightarrow sin\, \alpha L = 0$$

which happens only if $\alpha = \frac{n\pi}{L}$ for each $n = 0, 1, 2, \ldots$. Thus the solution becomes

$$U_n(x,t) = (B_n\, sin\, \frac{n\pi}{L} x) e^{-\alpha^2 c^2 t} \text{ for each } n.$$

and the general solution is obtained by summing up:

$$U(x,t) = \sum (B_n\, sin\, \frac{n\pi}{20} x) e^{-\alpha^2 c^2 t} \tag{30}$$

Imposing the third condition $U(x,0) = \frac{3}{2}x - 10$, we get

$$\frac{3}{2}x - 10 = \sum (B_n\, sin\, \frac{n\pi}{L} x) e^0,$$

which is a sine series. Hence B_n are given by

$$\begin{aligned} B_n &= \frac{2}{20} \int_0^L (\frac{3}{2}x - 10) sin \left(\frac{n\pi x}{L} \right) dx \\ &= -\frac{20}{n\pi} (1 + 2cos\, n\pi)\,. \end{aligned} \tag{31}$$

The solution to the problem is

$$U(x,t) = U_1(x) + U_2(x,t) = x + 40 - \sum (\frac{20}{n\pi} (1 + 2cos\, n\pi)) sin\, \frac{n\pi}{20} x) e^{-\alpha^2 c^2 t}$$

is the required solution.