Fourier Series Expansion

Deepesh K P

Suppose f is a real valued function from $\mathbb{R}$ to $\mathbb{R}$. In this note, we deal with the following three questions:

- When does f has a Fourier series expansion?
- How we find the expansion?
- What are the main properties of this expansion?

1 Existance of a Fourier series expansion:

There are three conditions which guarantees the existance of a valid Fourier series expansion for a given function. These conditions are collectively called the *Dirichlet conditions*:

1. f is a periodic function on $\mathbb{R}$. This means that there exists a period $T \geq 0$ such that
$$f(x) = f(x+T) \text{ for all } x \in \mathbb{R}.$$
2. f has only a finite number of maxima and minima in a period.
3. f has atmost a finite number of discontinuous points inside a period.

It should be noted that the second and third conditions are satisfied by almost all real valued functions that we deal with, inside any finite interval. But periodicity is a condition that is satisfied by very few functions like *constant function*, *sine*, *cos*, *tan* and their combinations. But we can consider any function defined on a finite interval $[a, b]$ (or (a, b)) as a periodic function on $\mathbb{R}$ by thinking that the function is extended to $\mathbb{R}$ by repeating the values in $[a, b]$ to the remaining part of $\mathbb{R}$. Thus

almost all functions (that we commonly use) defined on finite intervals can be expanded as Fourier series

Figure

2 Derivation of Fourier series expansion of a function defined in $[-\pi, \pi]$:

In Fourier series expansion, we would like to write the function as the sum of a series in sines and cosines in the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\, nx + b_n\, sin\, nx$$

For finding the above unknown co-efficients a_0, a_n and b_n in the Fourier series expansion of a function, one need to recall the value of certain integrals:

1. $\int_{-\pi}^{\pi} sin\, mx\, dx = 0$ for any integer m.
2. $\int_{-\pi}^{\pi} cos\, mx\, dx = 0$ for any integer m.
3. $\int_{-\pi}^{\pi} sin\, mx cos\, nx\, dx = 0$ for any integers m and n.
4. $\int_{-\pi}^{\pi} sin\, mx sin\, nx\, dx = 0$ for any integers $m \neq n$.
5. $\int_{-\pi}^{\pi} cos\, mx cos\, nx\, dx = 0$ for any integers $m \neq n$.
6. $\int_{-\pi}^{\pi} sin\, mx sin\, nx\, dx = \pi$ when the integers $m = n$.
7. $\int_{-\pi}^{\pi} cos\, mx cos\, nx\, dx = \pi$ when the integers $m = n$.

[All the above integrals easily follow by evaluating using integration by parts]

Now suppose $f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j cos\, jx + b_j\, sin\, jx$.

To find a_0:

Observe that

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\, dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} dx + \sum_{j=1}^{\infty} \left(a_j \int_{-\pi}^{\pi} cos\, jx\, dx + b_j \int_{-\pi}^{\pi} sin\, jx\, dx \right) \\ &= \frac{a_0}{2}\, 2\,\pi + \sum_{j=1}^{\infty} (0+0) \end{aligned}$$

This implies that

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

To find a_n:

Observe that

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) cos\, nx\, dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} cos\, nx\, dx + \sum_{j=1}^{\infty} \left(a_j \int_{-\pi}^{\pi} cos\, nx\, cos\, jx\, dx + b_j \int_{-\pi}^{\pi} cos\, nx\, sin\, jx\, dx \right) \\ &= \frac{a_0}{2}\, 0 + a_n\, \pi + \sum_{j=1}^{\infty} b_j\, 0 \end{aligned}$$

This implies that

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, cos\, nx\, dx$$

To find b_n:

Observe that

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) sin\, nx\, dx &= \frac{a_0}{2}\int_{-\pi}^{\pi} sin\, nx\, dx + \sum_{j=1}^{\infty}\left(a_j \int_{-\pi}^{\pi} sin\, nx\, cos\, jx\, dx + b_j \int_{-\pi}^{\pi} sin\, nx\, sin\, jx\, dx\right) \\ &= \frac{a_0}{2}\, 0 + \sum_{j=1}^{\infty} a_j\, 0 + b_n\, \pi.\end{aligned}$$

This implies that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sin\, nx\, dx$$

Thus

$$f(x) = \tfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\, nx + b_n\, sin\, nx,$$

where

$$a_0 = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, cos\, nx\, dx$$

$$b_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sin\, nx\, dx$$

[This expansion is valid at all those points x, where $f(x)$ is continuous.]

Note: Note that the above mentioned results hold when we take any $2\,\pi$ length intervals [This is because $\int_{c}^{c+2\pi} sin\, mx\, dx = 0, \ldots$ are true for any c].

So whenever we take a function f defined from $[c, c+2\pi]$ to $\mathbb{R}$, we have

$$f(x) = \tfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\, nx + b_n\, sin\, nx,$$

where

$$a_0 = \tfrac{1}{\pi}\int_{c}^{c+2\pi} f(x)\, dx$$

$$a_n = \tfrac{1}{\pi}\int_{c}^{c+2\pi} f(x)\, cos\, nx\, dx$$

$$b_n = \tfrac{1}{\pi}\int_{c}^{c+2\pi} f(x)\, sin\, nx\, dx$$

3 Derivation of Fourier series expansion of a function defined in $[a, b]$:

Now suppose $f(x)$ is defined in an arbitrary interval $[a, b]$. Let us define $\frac{b-a}{2} = l$, half the length of the interval. Now define $z = \frac{\pi}{l}x$. By this simple transformation, we can convert functions on any finite interval (say, $[a, b]$) to functions in the new variable z, whose domain is an interval of 2π length. This is because

$$x = a \Rightarrow z = \tfrac{\pi}{l}a \text{ and } x = b \Rightarrow z = \tfrac{\pi}{l}b = \tfrac{2\,\pi}{b-a}(b - a + a) = 2\,\pi + \tfrac{\pi}{l}a.$$

Thus when the variable x in $f(x)$ moves from a to b, the new variable z in the new function $F(z)$ (which is the same function f in the new variable) moves from c to $c+2\pi$, where $c = \frac{\pi}{l}a$. Hence the Fourier series expansion is applicable for $F(z)$. Thus

$$f(x) = F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\, nz + b_n\, sin\, nz,$$

where

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} F(z)\, dz$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} F(z)\, cos\, nz\, dz$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} F(z)\, sin\, nz\, dz$$

and changing back to the original variable x (note that $dz = \frac{\pi}{l}dx$), we have

$$f(x) = \tfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\, \frac{n\pi}{l}x + b_n\, sin\, \frac{n\pi}{l}x,$$

where

$$a_0 = \tfrac{1}{l}\int_a^b f(x)\, dx$$

$$a_n = \tfrac{1}{l}\int_a^b f(x)\, cos\, \frac{n\pi}{l}x\, dx$$

$$b_n = \tfrac{1}{l}\int_a^b f(x)\, sin\, \frac{n\pi}{l}x\, dx,$$

which is the general form of Fourier series expansion for functions on any finite interval. Also note that this is applicable to the first case of our discussion, where we need to take $a = -\pi$, $b = \pi$, $l = \pi$ and then everything becomes the same as in the initial stage.

3.1 Illustration

We now take a simple problem to demonstrate the evaluation of Fourier series.

Consider the function f defined by

$$f(x) = \begin{cases} -10 & \text{if } -2 \leq x \leq -1, \\ x & \text{if } -1 \leq x \leq 1, \\ 10, & \text{if } \quad 1 \leq x \leq 2. \end{cases}$$

We shall find the Fourier series expansion of this function. Here, note that the length of the interval is 4. So $2l = 4$ and $l = 2$. We need to write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\, \frac{n\pi}{2} x + b_n\, sin\, \frac{n\pi}{2} x,$$

where

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\, dx$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\, cos\, \frac{n\pi}{2} x\, dx$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\, sin\, \frac{n\pi}{2} x\, dx,$$

Now

$$\begin{aligned} a_0 &= \frac{1}{2}(\int_{-2}^{-1} -10\, dx + \int_{-1}^{1} x\, dx + \int_{1}^{2} 10\, dx) \\ &= \frac{1}{2}(-10 + 0 + 10) = 0 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{2}(\int_{-2}^{-1} -10\, cos\, \frac{n\pi}{2} x\, dx + \int_{-1}^{1} -10\, cos\, \frac{n\pi}{2} x\, dx + \int_{1}^{2} -10\, cos\, \frac{n\pi}{2} x\, dx) \\ &= \frac{1}{2}(-10 \int_{-2}^{-1} cos\, \frac{n\pi}{2} x\, dx + \int_{-1}^{1} x cos\, \frac{n\pi}{2} x\, dx + 10 \int_{1}^{2} cos\, \frac{n\pi}{2} x\, dx) \\ &= \frac{1}{2}(\frac{20}{n\pi}(sin\, \frac{n\pi}{2} - sin\, n\pi) + 0 + \frac{20}{n\pi}(sin\, n\pi - sin\, \frac{n\pi}{2})) = 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{2}(\int_{-2}^{-1} -10\, sin\, \frac{n\pi}{2} x\, dx + \int_{-1}^{1} x\, sin\, \frac{n\pi}{2} x\, dx + \int_{1}^{2} 10\, sin\, \frac{n\pi}{2} x\, dx) \\ &= \frac{1}{2}\{\frac{20}{n\pi}(cos(\frac{n\pi}{2}) - cos(n\pi)) + 2[\frac{-2}{n\pi} cos(\frac{n\pi}{2}) + \frac{4}{n^2\pi^2} sin(\frac{n\pi}{2}) - 0] - \frac{20}{n\pi}(cos(n\pi) - cos(\frac{n\pi}{2})\} \\ &= \frac{18}{n\pi} cos(\frac{n\pi}{2}) - \frac{20}{n\pi} cos(n\pi) + \frac{4}{n^2\pi^2} sin(\frac{n\pi}{2}). \end{aligned}$$

So when $n = 1 \Rightarrow b_1 = \frac{4}{\pi^2},\ n = 2 \Rightarrow b_2 = \frac{19}{\pi}, \ldots$

Thus the Fourier expansion of $f(x)$ is

$$\begin{aligned} f(x) &= \frac{0}{2} + 0\, cos\, \frac{\pi}{2}x + \frac{4}{\pi^2} sin\, \frac{\pi}{2}x + 0\, cos\, \frac{2\pi}{2}x + \frac{19}{\pi}\, sin\, \frac{2\pi}{2}x + \dots \\ &= \frac{4}{\pi^2} sin\, \frac{\pi}{2}x + \frac{19}{\pi}\, sin\, \frac{2\pi}{2}x + \dots, \end{aligned}$$

which is valid at all points in $[-2, 2]$ except at -1 and 1, since the function is continuous at all points except -1 and 1.

3.2 Some special cases:

Suppose the function is an odd/even function in a symmetric interval $[-c, c]$. That is $f(-x) = f(x)$ for all $x \in \mathbb{R}$ [$\Rightarrow$ Even] or $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ [$\Rightarrow$ Odd]. Then from the results,

$$\int_{-c}^{c} f(x)\, dx = 0 \text{ when } f(x) \text{ is odd}$$

$$\int_{-c}^{c} f(x)\, dx = 2 \int_{0}^{c} f(x)\, dx \text{ when } f(x) \text{ is even },$$

we have some specialities in the expansions.

3.2.1 Even functions:

Suppose $f(x)$ is even in a domain $[-c, c]$. Then it can be observed that $b_n = 0$ and so the Fourier series becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\, \frac{n\pi}{c} x$$

where

$$a_0 = \frac{2}{c} \int_{0}^{c} f(x)\, dx$$

$$a_n = \frac{2}{c} \int_{0}^{c} f(x)\, cos\, \frac{n\pi}{c} x\, dx$$

3.2.2 Odd functions:

Similarly when $f(x)$ is odd in a domain $[-c, c]$. Then $a_0 = a_n = 0$ and the Fourier series becomes

$$f(x) = \sum_{n=1}^{\infty} b_n sin\, \frac{n\pi}{c} x$$

where

$$b_n = \frac{2}{c} \int_{0}^{c} f(x)\, sin\, \frac{n\pi}{c} x\, dx$$

Note: If you observe carefully, in the illustration problem, the function is actually odd. Note that the Fourier series contains only sine terms. But the converse is not true in general. That is, even if the Fourier series contains only sine terms, the function may not be odd!

[For proving the above two cases, one should recall that the product of an odd and an even function is always odd and when both functions are even or both odd then the product is always even.]

3.3 Evaluation of series

Fourier series can be used for evaluating the sum of certain series. For each value of $f(x_0)$, where x_0 is a continuous point of the function, we get a series by putting the value x_0 on both sides of the function.

Illustration

Suppose $f(x) = x^2$, $-\pi < x < \pi$. We shall find the Fourier series of this function and use it to evaluate the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$.

Observe that the function is an even function in the symmetric interval $(-\pi, \pi)$. So we immediatly get $b_n = 0$ for all n. So we need to find only $a_o, \& a_n$.

$$a_0 = \int_{-\pi}^{\pi} x^2\, dx = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, cos\, nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} x^2\, cos\, nx\, dx = (-1)^n \frac{4}{n^2}$$

Thus the Fourier series becomes

$$x^2 = \frac{\pi^2}{3} + -1.\frac{4}{1^2} cos\, x + 1.\frac{4}{2^2} cos\, 2x + -1.\frac{4}{3^2} cos\, 3x + \ldots$$

Now to find the series sum, take $x = 0$, which is a continuous point. Substituting on both sides

$$0 = \frac{\pi^2}{3} + -4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\right),$$

which gives

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots = \frac{\pi^2}{12}.$$

4 Fourier Sine Series and Cosine Series:

Sometimes we may need to expand a given function as a series in only cosine or only sine terms. Such series are called cosine series and sine series respectively. This can be done easily when the given function is defined on an interval like $[0, c]$ (that is an interval on the positive side - negative side is also possible). Hence such series are also reffered as *Half range series expansion.*

Suppose $f(x)$ is defined in the interval $[0, c]$. Then one can think of the extended function as either even or odd (depending on the need; i.e., cosine series/sine series) in the interval $[-c, c]$.

Extension as even function - Cosine series:

Suppose we think of $f(x)$ as even in the interval $[-c, c]$. Now expand this function. Obviously the series you obtain will have only constant and cosine terms. This is the cosine series of the function $f(x)$. That is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\,\frac{n\pi}{c}x$$

where

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi}{c}x\,dx$$

Extension as odd function - Sine series:

Similarly when $f(x)$ is thought to be extended as an odd function in a domain $[-c, c]$. Then the Fourier series becomes

$$f(x) = \sum_{n=1}^{\infty} b_n sin\,\frac{n\pi}{c}x$$

where

$$b_n = \frac{2}{c}\int_0^c f(x)\,sin\,\frac{n\pi}{c}x\,dx$$

Note: For doing problems one need not always do the extension to $[-a, a]$. Just applying the formula is fine.

Note: A function defined in $[0, a]$ will have a Fourier series expansion, a Cosine series expansion and a Sine series expansion.

Illustration

Consider the function $f(x) = x$, $0 < x < \pi$. We shall find the Fourier cosine series of this function.

First, considering as an even function in $(-\pi, \pi)$, we get the Fourier cosine series as $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$

Note that $l = \pi$ here. Let us find a_0 and a_n.

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} x\,dx \\ &= \frac{2}{\pi}\frac{\pi^2}{2} = \pi^2. \\ a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx \\ &= \frac{2}{\pi}\left[\frac{(-1)^2}{n^2} - \frac{1}{n^2}\right] \end{aligned}$$

Thus the Fourier cosine series is given by

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[cos\,x + \frac{cos\,3x}{3^2} + \frac{cos\,5x}{5^2} + \ldots\right]$$

In a similar way, one can apply the formula to find the Fourier sine series of the function. Note that the same function also has got a Fourier series expansion, where you need to apply the formula by taking $l = \frac{\pi}{2}$.

5 Practical Harmonic Analysis

In many practical problems, we get the data at discrete points of time. That is, the function will not be given explicitly but the function values will be given at, say, regular intervals of the domain. For example,

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$3\frac{\pi}{4}$	π
$f(x)$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	-1

Here the function is actually $sin\, x$, but given at discrete points, 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $3\,\frac{\pi}{4}$, π. With these point values, how to calculate the Fourier series?

5.1 The Trapezoidal Rule of Integration

This is a rule used for finding the definite integral of a function, which is given at equally spaced, discrete points. Suppose the values of a function are given as in

x	x_0	x_1	x_2	$\ldots$	x_{n-1}	x_n
$f(x)$	y_0	y_1	y_2	$\ldots$	y_{n-1}	y_n

were the points $x_0, x_1, \ldots, x_n$ are equally spaced; i.e., $x_{i+1} - x_i = h$, a constant, for all i. Then the Trapezoidal rule says that

$$\int_{x_0}^{x_n} f(x)\, dx = \frac{h}{2}[y_0 + 2\,(y_1 + y_2 + \ldots + y_{n-1}) + y_n]$$

5.1.1 Example

Calculate $\displaystyle\int_1^4 f(x)dx$ where $f(x)$ is given at

x:	1	1.5	2	2.5	3	3.5	4
f(x):	2	4.875	10	18.125	30	46.375	68

Solution: Note that $h = 0.5$ here. Now from trapezoidal rule,

$$\int_1^4 f(x)dx = \frac{0.5}{2}\,(2 + 2[4.875 + 10 + 18.125 + 30 + 46.375] + 68) = 72.1875.$$

5.2 Calculating the Fourier series from a discrete data

Suppose that the values of a function are given as in

x	x_0	x_1	x_2	$\ldots$	x_{n-1}	x_n
$f(x)$	y_0	y_1	y_2	$\ldots$	y_{n-1}	y_n

Here the length of the interval is $x_n - x_0$. So $l = \frac{x_n - x_0}{2}$. Hence the Fourier series of the function is

$$
\boxed{\begin{array}{l}
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\,\frac{n\pi}{l}x + b_n\,sin\,\frac{n\pi}{l}x, \\
\text{where} \\
\quad a_0 = \frac{1}{l}\int_{x_0}^{x_n} f(x)\,dx \\
\quad a_n = \frac{1}{l}\int_{x_0}^{x_n} f(x)\,cos\,\frac{n\pi}{l}x\,dx \\
\quad b_n = \frac{1}{l}\int_{x_0}^{x_n} f(x)\,sin\,\frac{n\pi}{l}x\,dx,
\end{array}}
$$

Since $f(x)$ is not explicitly given here, we need to use the Trapezoidal rule to find $a_0, a_n \,\&\, b_n$:

$$
\begin{aligned}
a_0 &= \frac{1}{l}\int_{x_0}^{x_n} f(x)\,dx \\
&= \frac{1}{l}\frac{h}{2}[y_0 + 2\,(y_1 + y_2 + \ldots + y_{n-1}) + y_n] \\
a_n &= \frac{1}{l}\int_{x_0}^{x_n} f(x)\,cos\,\frac{n\pi}{l}x\,dx \\
&= \frac{1}{l}\frac{h}{2}\left[y_0\,cos\frac{2n\pi}{h}x_0 + 2\left(y_1\,cos\frac{2n\pi}{h}x_1 + y_2\,cos\frac{2n\pi}{h}x_2 + \ldots + y_{n-1}\,cos\frac{2n\pi}{h}x_{n-1}\right) + y_n\,cos\frac{2n\pi}{h}x_n\right] \\
b_n &= \frac{1}{l}\int_{x_0}^{x_n} f(x)\,sin\,\frac{n\pi}{l}x\,dx \\
&= \frac{1}{l}\frac{h}{2}\left[y_0\,sin\frac{2n\pi}{h}x_0 + 2\left(y_1\,sin\frac{2n\pi}{h}x_1 + y_2\,sin\frac{2n\pi}{h}x_2 + \ldots + y_{n-1}\,sin\frac{2n\pi}{h}x_{n-1}\right) + y_n\,sin\frac{2n\pi}{h}x_n\right]
\end{aligned}
$$

In problems, you can not find all the a_ns. So find a_0, a_1, a_2, b_1, b_2 and substitute in the Fourier series formula.

Illustration

Find the Fourier series of the function from the data given below:

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$3\frac{\pi}{4}$	π
$f(x)$	1	2	3	4	5

Here the interval under consideration is $(0, \pi)$, which is of length π. Hence $l = \frac{\pi}{2}$ and $h = \frac{\pi}{4}$. The Fourier series of the function is

$$
\boxed{\begin{array}{l}
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\,2nx + b_n\,sin\,2nx, \\
\text{where} \\
\quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \\
\quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,cos\,2nx\,dx \\
\quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,sin\,2nx\,dx,
\end{array}}
$$

Now we calculate a_0, a_1, a_2, b_1, b_2 using the Trapezoidal rule.

$$\begin{aligned}
a_0 &= \frac{1}{\frac{\pi}{2}}\int_0^{\pi} f(x)\,dx \\
&= \frac{2}{\pi}\left\{\frac{\pi}{8}\left[1+2\,(2+3+4)+5\right]\right\} = 6 \\
a_1 &= \frac{1}{\frac{\pi}{2}}\int_0^{\pi} f(x) cos\, 2x\,dx \\
&= \frac{2}{\pi}\left\{\frac{\pi}{8}\left[1\,cos\,2\,.0+2\,\left(2\,cos\,2.\frac{\pi}{4}+3\,cos\,2.\frac{\pi}{2}+4\,cos\,2.3\frac{\pi}{4}\right)+5\,cos\,2\pi\right]\right\} \\
&= \frac{1}{4}\left[1+2\;(2.0+3.-1+4.0)+5.1\right] = 0 \\
a_2 &= \frac{1}{\frac{\pi}{2}}\int_0^{\pi} f(x) cos\, 4x\,dx \\
&= \frac{2}{\pi}\left\{\frac{\pi}{8}\left[1\,cos4\,.0+2\,\left(2\,cos\,4.\frac{\pi}{4}+3\,cos\,4.\frac{\pi}{2}+4\,cos\,4.3\frac{\pi}{4}\right)+5\,cos\,4\pi\right]\right\} \\
&= \frac{1}{4}\left[1+2\;(2.-1+3.1+4.-1)+5.1\right] = 0 \\
b_1 &= \frac{1}{\frac{\pi}{2}}\int_0^{\pi} f(x) sin\, 2x\,dx \\
&= \frac{2}{\pi}\left\{\frac{\pi}{8}\left[1\,sin\,2\,.0+2\,\left(2\,sin\,2.\frac{\pi}{4}+3\,sin\,2.\frac{\pi}{2}+4\,sin\,2.3\frac{\pi}{4}\right)+5\,sin\,2\pi\right]\right\} \\
&= \frac{1}{4}\left[0+2\;(2.1+3.0+4.-1)+5.0\right] = -1 \\
b_2 &= \frac{1}{\frac{\pi}{2}}\int_0^{\pi} f(x) sin\, 4x\,dx \\
&= \frac{2}{\pi}\left\{\frac{\pi}{8}\left[1\,sin4\,.0+2\,\left(2\,sin\,4.\frac{\pi}{4}+3\,sin\,4.\frac{\pi}{2}+4\,sin\,4.3\frac{\pi}{4}\right)+5\,sin\,4\pi\right]\right\} \\
&= \frac{1}{4}\left[1.0+2\;(2.0+3.0+4.0)+5.0\right] = 0
\end{aligned}$$

Thus the Fourier series expansion of the function is

$$f(x) = \frac{0}{2} + 0.cos\,2x + 0.cos\,4x + \ldots + -1.sin\,2x + 0.sin\,4x + \ldots$$

6 Problems

1. Obtain the Fourier series representation of $f(x) = \frac{1}{4}(\pi - x)^2,\;\; 0 < x < 2\pi$.
   [Hint: $l = \pi,\; a_0 = \frac{\pi^2}{6},\; a_n = \frac{1}{n^2},\; b_n = 0$]
2. Expand $f(x) = x\,sin\,x,\;\; 0 \le x \le 2\pi$ as a Fourier series.
   [Hint: Even, $l = \pi,\; b_n = 0,\; a_0 = -2,\; a_n = \frac{2}{n^2-1},$ if $n \neq 1,\; a_1 = \frac{-1}{2}$]